United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 2 : Thursday, 26 January 2012

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1, 2, 3, 4 in order, and the cover sheet at the front.*
- To accommodate candidates sitting in other timezones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 27 January.

In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (29 March – 2 April 2012). At the training session, students sit a pair of IMO-style papers and eight students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of six for this summer's IMO (to be held in Mar del Plata, Argentina, 4–16 July) will then be chosen.

Do not turn over until **told to do so.**

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2011/12 British Mathematical Olympiad
Round 2

1. The diagonals AC and BD of a cyclic quadrilateral meet at E. The midpoints of the sides AB, BC, CD and DA are P, Q, R and S respectively. Prove that the circles EPS and EQR have the same radius.

2. A function f is defined on the positive integers by $f(1) = 1$ and, for $n > 1$,
$$f(n) = f\left(\left\lfloor \frac{2n-1}{3} \right\rfloor\right) + f\left(\left\lfloor \frac{2n}{3} \right\rfloor\right)$$
where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Is it true that $f(n) - f(n-1) \le n$ for all $n > 1$?

   [Here are some examples of the use of $\lfloor x \rfloor$: $\lfloor \pi \rfloor = 3$, $\lfloor 1729 \rfloor = 1729$ and $\lfloor \frac{2012}{1000} \rfloor = 2$.]

3. The set of real numbers is split into two subsets which do not intersect. Prove that for each pair (m, n) of positive integers, there are real numbers $x < y < z$ all in the same subset such that $m(z-y) = n(y-x)$.

4. Show that there is a positive integer k with the following property: if a, b, c, d, e and f are integers and m is a divisor of
$$a^n + b^n + c^n - d^n - e^n - f^n$$
for all integers n in the range $1 \le n \le k$, then m is a divisor of $a^n + b^n + c^n - d^n - e^n - f^n$ for all positive integers n.